Eur. J. Mech. B/Fluids

THE EQUILIBRIUM RANGE CASCADES OF WIND-GENERATED WAVES

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(Received 22 September 1998, revised and accepted 2 February 1999)

Abstract – It was established theoretically by Zakharov and Filonenko (1966) that the direct cascade region of the equilibrium range of the spectrum should follow an f^{-4} variation. This has since been verified from experimental data, by Toba (1973), Donelan et al (1985) and others. In this study we present a numerical verification of this f^{-4} variation, assuming physically realistic parameterizations for nonlinear wave-wave interactions, S_{nl}, for energy input to waves by the wind, S_{in} and removed by wave - breaking dissipation, S_{ds}. © Elsevier, Paris

1. Introduction

Ocean surface waves are generated, grow, evolve and are dissipated according to the spectral wave energy equation

$$\frac{\partial E(f,\theta)}{\partial t} + \underset{\sim}{C}_g \cdot \nabla E(f,\theta) = S_{nl} + S_{in} + S_{ds} \tag{1}$$

where $f, \theta, \vec{x}$ and t represent frequency, direction, position and time, respectively. Spectral wave energy is represented by $E(f,\theta)$ and wave group velocity is $\underset{\sim}{C}_g$. Nonlinear wave-wave interactions are represented by S_{nl}. The energy input to the waves from the wind is represented by S_{in} and the dissipation due to wave-breaking is given by S_{sds}.

There are several challenges that must be confronted in using equation (1) to simulate wave growth and development. For example, although several decades have been invested in the construction of realistic formulations for wind input S_{in} and dissipation S_{ds}, the state-of-the-art parameterizations for these terms still involve many assumptions. This is described in Komen et al. (1994). Although accepted formulations for S_{in} and S_{ds} may be valid for specific field experiments (i.e. the Bight of Abaco experiment by Snyder et al. 1981), these parameterizations are not generally valid for more general situations. Examples of the latter conditions are intense, rapidly varying storms with complex spatially and temporally varying wind stress. Even in simple constant wind conditions there is evidence that parameterizations for S_{in} and S_{ds} are less than perfect. For example, the accepted parameterizations for S_{in}, S_{ds} and S_{nl}, as implemented in the WAM model of Komen et al. (1994), give energy growth curves that differ significantly from observations for short fetches, as shown by Tolman and Chalikov (1996).

The equilibrium range of the spectrum is above the spectral peak, f_p, extending from about $1.5 \times f_p$ to about $3 \times f_p$. In this region there is an approximate dynamical balance between the so-called direct transfer of energy from lower to higher frequencies, and the so-called inverse transfer of energy from high to lower frequencies. At a specific critical frequency f_c in the equilibrium range, the direct and inverse transfers balance. Below f_c, there is a net transfer of energy to lower frequencies. In this region, inverse cascades occur. Above f_c, there is a net transfer of energy to high frequencies. In this region, direct cascades occur. Following Zakharov (1991), one can associate the direct cascades with wave dissipation S_{ds} and consequent wave attenuation. Inverse cascades can be associated with wind input, S_{in}, and wave growth. Clearly, inaccuracies in parameterizations for S_{in} and S_{ds} will be reflected by inaccuracies in estimates for the variation of the equilibrium range with frequency. Zakharov (1991) suggested that the direct cascade region follows an f^{-4} variation, and the inverse cascade region, an $f^{-\frac{11}{3}}$ variation. This paper is concerned with finding the parameterizations for S_{in} and S_{ds} which are consistent with an f^{-4} variation for the direct cascade region of the equilibrium range, as originally predicted by Zakharov and Filonenko (1966), observed by Toba (1973), and reviewed by Zakharov (1991).

In Section 2, we present a brief review of accepted "state-of-the-art" formulations for S_{in} and S_{ds}, for example as implemented in the WAM model of Hasselmann et al. (1988) and Komen et al. (1994). In Section 3, we describe accepted parameterizations of observed spectra, such as Battjes - JONSWAP, as derived by Battjes et al. (1987), and the formulations for S_{in} and S_{ds} which may be derived from these parameterizations. In Section 4, we consider these S_{in} and S_{ds} parameterizations, and we show that they can lead to the existence of equilibrium ranges which have an approximate f^{-4} variation, with regions of direct and inverse cascades that appear qualitatively consistent with the theoretical estimates of Zakharov and Filonenko (1966) and Zakharov (1991).

2. Overview of S_{in} and S_{ds} Parameterizations

2.1. *WAM Model Formulation*

The accepted formulation for wind input S_{in}, as parameterized in the WAM model of Hasselmann et al. (1988) and Komen et al. (1994), suggests that S_{in} should be represented as

$$S_{in}(f,\theta) = \beta\omega E(f,\theta), \tag{2}$$

where β is a nondimensional function of seastate maturity and $\omega = 2\pi f$ represents angular frequency, which is related to wavenumber k through the deep water dispersion relation $\omega^2 = gk$.

In earlier versions, for example WAM (cycle 3), as described by Hasselmann et al. (1988), the formulation for β is given by

$$\beta = \max\left\{0, 0.25\epsilon\left(28\frac{U_*}{C_p}\cos(\Delta\theta) - 1\right)\right\} \tag{3}$$

where $\epsilon = \frac{\rho_a}{\rho_w}$ is the ratio of the densities of air and water and $\Delta\theta =$ the difference between the wind and wave directions.

Janssen (1989,1991), Komen et al. (1994), Tolman and Chalikov (1996) have made revisions to β, aimed at simulating the coupling feedback between wind and waves. These modifications have been able to simulate (i) negative wind input when the wind is at large angles to the waves, or slower than the waves, and (ii), smaller overall energy input, because of negative contributions for overdeveloped waves and smaller contributions for waves near full development.

The accepted formulation for wave dissipation S_{ds}, motivated by the earlier studies by Hasselmann (1974) and Komen et al. (1984), as parameterized in the WAM model of Hasselmann et al. (1988) and Komen et al. (1994), suggests that S_{ds} should have the form,

$$S_{ds} = C_{ds}\left(\frac{\hat{\alpha}}{\hat{\alpha}_{PM}}\right)^2\left(\frac{\omega}{\bar{\omega}}\right)^2\bar{\omega}E(f,\theta) \tag{4}$$

where $\hat{\alpha} = m_0\bar{\omega}^4/g^2$, and m_0 is the zeroth moment of variance spectrum, $\bar{\omega}$ is the mean radial frequency $\iint E(\omega,\theta)\omega \, d\omega \, d\theta / E_{total}$, with E_{total} the total spectral energy and f is the frequency in Hz. Tuning is achieved by the fitting parameter, C_{ds}, and $(\hat{\alpha}/\hat{\alpha}_{PM})$ is an overall measure of steepness in the wave field.

Variations in this S_{ds} formulation are given by Banner and Young (1994) and Tolman and Chalikov (1996). These alternate S_{ds} formulations result from alternate choices for the other source terms: S_{in} and S_{nl}. Banner and Young (1994) consider the nonlinear interactions S_{nl} as simulated by Resio and Perrie (1991). Tolman and Chalikov (1996) implement a very detailed, physically well-motivated, wind input S_{in} parameterization. It is shown in Lin and Perrie (1999) that the S_{nl} formulation by Resio and Perrie (1991) is comparable to the EXACT-NL code of Hasselmann and Hasselmann (1981), and both differ from the DIA code of Komen et al. (1994) in the WAM model. As stated by Komen et al. (1994), all S_{ds} formulations are approximations based on an incomplete understanding of wave - breaking, and represent a tuning of the computer code, so that the resultant simulation of equation (1) produces energy growth and evolution consistent with accepted observations, such as the JONSWAP data of Hasselmann et al. (1973) and Battjes et al. (1987).

2.2. *Empirical Formulation*

The JONSWAP wave spectra was given a parameterization by Hasselmann et al. (1973) which may be written as

$$E(f) = \alpha_5 g^2 (2\pi)^{-4} f^{-5} \exp^{\frac{-5}{4}\left(\frac{f}{f_p}\right)^{-4}} \gamma^{exp\left[-\left(\frac{f-f_p}{2\sigma f_p}\right)^2\right]} \tag{5}$$

where $E(f)$ is the one - dimensional wave energy, γ is the peak enhancement factor, α_5 is the spectral level of the tail, and σ_a and σ_b represent the peak asymmetry, where

$$\sigma = \begin{cases} \sigma_a & \text{for } f \leq f_p \\ \sigma_b & \text{for } f > f_p. \end{cases} \tag{6}$$

Hasselmann et al. (1973) gave parameterizations for f_p, γ, α_5, σ_a and σ_b as functions of dimensionless fetch, xg/U_*^2.

Some years later, Battjes et al. (1987), showed the JONSWAP data is better described by an f^{-4} fit, using the parameterization,

$$E(f) = \alpha_4 g U_* (2\pi)^{-3} f^{-4} \exp^{\left(-\left(\frac{f}{f_p}\right)^{-4}\right)} \gamma^{exp\left[-\left(\frac{f-f_p}{2\sigma f_p}\right)^2\right]}. \tag{7}$$

Fetch-relations for f_p, γ, α_4, σ_a and σ_b, differ significantly from those derived by Hasselmann et al. (1973).

Direct and inverse cascades can be computed in terms of the direct and inverse action transfers, following Resio and Perrie (1991). For example, inverse transfers, Γ_A^- may be written as

$$\Gamma_A^-(\omega_r) = \int\int\oint C^2 \mathcal{D} \left|\frac{\partial \mathcal{W}}{\partial n}\right|^{-1} H(|\, \underline{k_3} \,| - \underline{k_r}) H(\underline{k_r} - |\, \underline{k_1} \,|) ds d\underline{k_1} d\underline{k_3} \tag{8}$$

where C^2 the coupling coefficient, $|\frac{\partial \mathcal{W}}{\partial n}|^{-1}$ is the Jacobian for the coordinate transformation, resonance conditions are $\underline{k}_1 + \underline{k}_2 + \underline{k}_3 + \underline{k}_4 = 0$ and $\omega_1 + \omega_2 + \omega_3 + \omega_4 = 0$, where $ds d\underline{k}_1 d\underline{k}_3$ is transformed phase space volume, H is the Heaviside function, and the density function is $D(\underline{k}_1, \underline{k}_2, \underline{k}_3, \underline{k}_4) = (n(\underline{k}_1) + n(\underline{k}_4))\, n(\underline{k}_2) n(\underline{k}_3) - (n(\underline{k}_3) + n(\underline{k}_2))\, n(\underline{k}_1) n(\underline{k}_4)$.

Using the energy spectra parameterized in equations (5)-(7), Perrie and Lin (1997, 1999) use direct calculations of the direct and inverse cascades from equation (8) to infer constraints on the functional forms for S_{in}

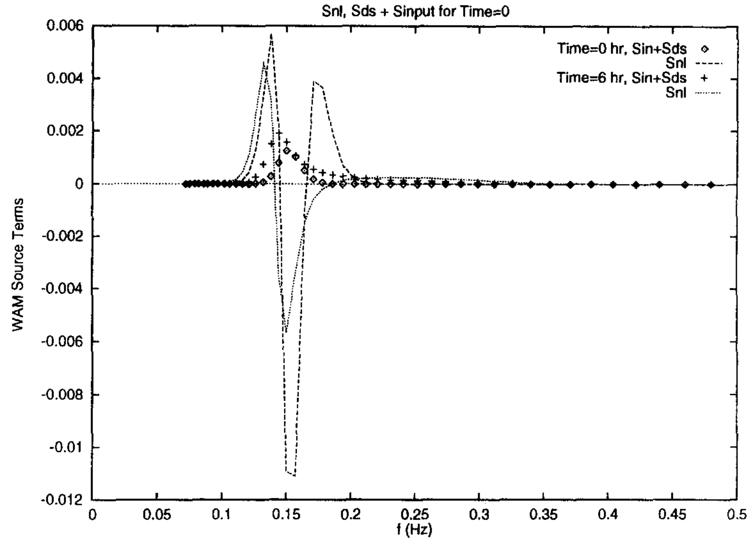

FIGURE 1. The variation of S_{nl} and $S_{in} + S_{ds}$ with frequency f, at time $t = 0hr$ and $t = 6hr$, where S_{in} and S_{ds} are WAM source terms scaled by 1/100.

and S_{ds}. Motivated by the WAM formulations of equations (2)-(4), which suggest that S_{in} and S_{ds} have the generic form

$$S_{in} + S_{ds} \sim E(f) f^m \tag{9}$$

where $m \approx 2$, they suggest that $m \sim$ spectral maturity. Specifically, they suggest that

$$m = a_0 + a_1(f_p - A) + a_2(f_p - B)^2 \tag{10}$$

where a_0, a_1, a_2, A and B are appropriate constants.

3. Equilibrium Range Calculation

3.1. *WAM Source Terms*

Using the WAM parameterizations for S_{in} and S_{ds}, we integrate the kinetic equation (1) in time, neglecting the spatial advective term $\underline{C}_g \cdot \nabla E(f, \theta)$. These S_{in} and S_{ds} formulations in equations (2)-(4) are tuned to the discrete interaction approximation (DIA), in the standard WAM model. Because DIA differs significantly from more accurate simulations of the nonlinear wave-wave transfer S_{nl}, such as that of Resio and Perrie (1991), which is used here, it is necessary to rescale these WAM formulations for S_{in} and S_{ds} in order to get physically realistic results. Thus we reduce S_{in} and S_{ds} by a factor of 100. Note that this (1/100) scaling gives very small values for $S_{in} + S_{ds}$, relative to maximal values for S_{nl}.

This test uses a fine-mesh grid, with 58 frequency bands, 2.5° angular resolution and 15 s timesteps. Initially, f_p is 0.15 Hz and f_c, the interface between direct and inverse cascades, is about 0.22 Hz. Assumed wind speed is 15 m/s. After 6 hr simulation the spectral peak f_p has migrated to lower frequency values and all variables appear physically realistic, as shown in Figure 1. The critical frequency is about 0.17 Hz. In Figure 2, we show the exponential variation of the high frequency region of the spectrum. This shows that starting with the f^{-5} variation assumed by the JONSWAP parameterization of Hasselmann et al. (1973), the spectrum evolves to having a distinct equilibrium range at about $1.8f_p \lesssim f \lesssim 2.4f_p$. The direct cascade region of the

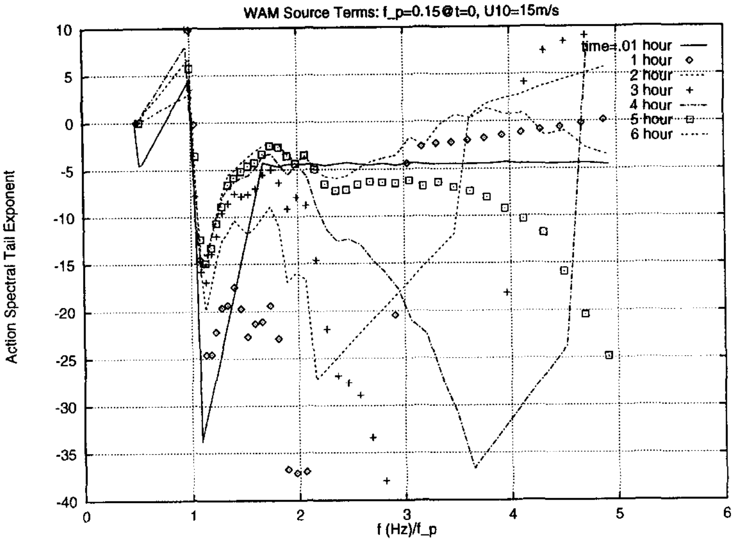

FIGURE 2. The variation of the equilibrium range exponent m, where $E(f) \sim f^m$ as a function of normalized frequency f/f_p, with time, up to $t = 6hr$, for the spectral evolution of Figure 1.

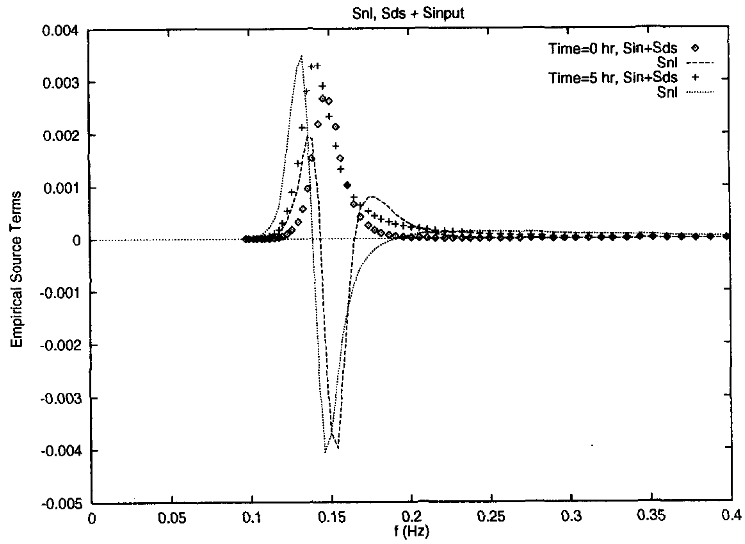

FIGURE 3. As in Figure 1, using the empirical source terms S_{in} and S_{ds} of equations (9)-(10).

equilibrium range has a variation of about f^{-4}. As suggested by Zakharov (1991), the inverse cascade region of the equilibrium range, below f_c, has a slightly smaller equilibrium range exponent, in magnitude.

3.2. *Empirical S_{in} and S_{ds}*

Using the empirical $S_{in} + S_{ds}$ parameterizations of equations (9)-(10), we repeated the calculation of the previous section, as given in Figures 3-4. Starting with an initial $f_p \approx 0.15$ and $f_c \approx 0.18$, we simulate the source terms, $S_{in} + S_{ds}$ and S_{nl} for $15hr$. It is interesting to note the magnitude of $| S_{in} + S_{ds} |$ relative to $| S_{nl} |$ in Figure 3, as compared to the WAM test in Figure 1. Similar magnitudes for WAM S_{in} and S_{ds} terms would give unphysical results.

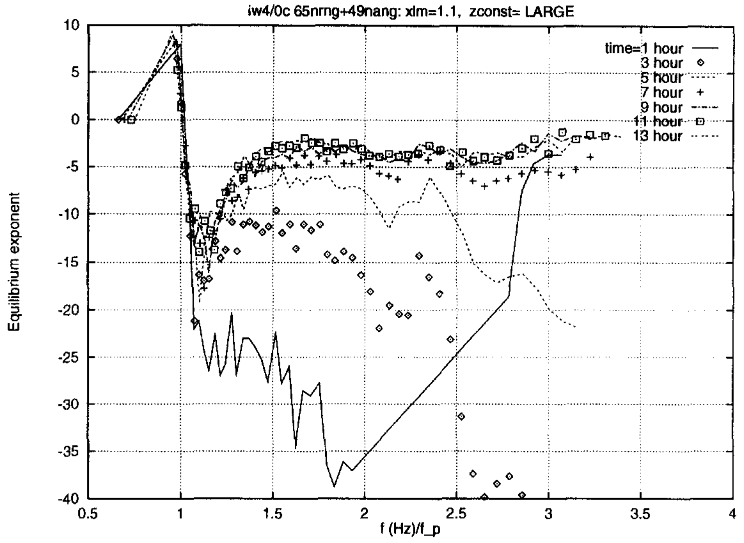

FIGURE 4. As in Figure 2, showing the variation in m as a function of normalized frequency f/f_p, with time, up to $t = 13hr$.

Figure 4 gives the evolution of the equilibrium range exponent across the spectrum, for this $S_{in} + S_{ds}$ parameterization, as in Figure 2. Compared to the WAM formulations for S_{in} and S_{ds} in Figures 1-2, the equilibrium range is much broader and better defined. Figure 4 shows that after a few hours, the direct and inverse cascade regions are formed in the domain $1.4f_p \lesssim f \lesssim 2.8f_p$. From computations simulating longer times than those of Figure 3, we estimate that f_c, marking the interface between direct and inverse cascade regions, is about $2f_p$. Therefore the equilibrium range variation for the direct cascade region of the spectrum, above f_c is approximately f^{-4}. For the inverse cascade region, below f_c, the exponent of the equilibrium range variation is slightly smaller in magnitude, which is consistent with Zakharov (1991).

4. Conclusions

The nonlinear wave-wave interactions are responsible for the formation of the equilibrium range, including its direct and inverse cascade regions, with their f^{-4} and $f^{-\frac{11}{3}}$ variations, (Zakharov and Filonenko: 1966 and Zakharov: 1991). A simulation of the kinetic equation (1), using an accurate realistic S_{nl} always gives a well-behaved equilibrium range, a critical frequency f_c, defining the interface between direct cascades to high frequencies and inverse cascades to lower frequencies. As pointed out by Zakharov (1991), such dynamical structure is necessary, for conservation of energy and action within the spectrum.

Assuming the existence of nonzero source terms for wind input S_{in} and dissipation S_{ds} makes the problem difficult because it is easy to specify unphysical S_{in} and S_{ds} formulations which give unrealistic results. It is difficult to guess physically well-motivated formulations for S_{in} and S_{ds} which are also consistent with equilibrium range dynamics. Using the S_{nl} formulation of Resio and Perrie (1991), the WAM formulations for S_{in} and S_{ds} in their normal form lead to unrealistic effects, after simulations of only a few hours, because WAM S_{in} and S_{ds} are tuned to DIA, the parameterization for the WAM S_{nl}.

With more accurate formulations for S_{nl}, new formulations for S_{in} and S_{ds} are necessary in order to simulate wave growth relations in a realistic manner. This was the finding of Banner and Young (1994). We have shown that suitable rescaling of WAM S_{in} and S_{ds} parameterizations can result in a small credible equilibrium range, with associated regions of direct and inverse cascades. However, there are other parameterizations for S_{in} and S_{ds} which can result in a realistic equilibrium range and regions of direct and inverse cascades. This was shown in Figures 3-4, for the empirical $S_{in} + S_{ds}$ formulation of Perrie and Lin (1997, 1999). This formulation was derived from the JONSWAP data, as parameterized by Hasselmann et al. (1973) and Battjes et al. (1987).

Acknowledgements

The wave modeling program at BIO is funded by the Federal Panel on Energy Research and Development (Canada) under Project 534201. This research was also funded by the Wave Prediction Base Enhancement Program of the Office of Naval Research and by the ONR grant N 00014-98-1-0070.

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